Section 8.6

Integration Strategies

We have undoubtedly noticed that integration is more of a challenge than differentiation. So how does one approach integration of a function that may arise while working on some application? Numerical methods exist, which are how computer systems approximate definite integrals, but it is not difficult to find an integral which numerical methods struggle with if we ask for a high enough degree of accuracy in the approximation.

There really is no way around the necessity of strategies to solve integrals analytically, or at the very least reduce them to something that is manageable by computer systems.

General Strategy for Integration

1. **Simplify the Integrand if Possible.** Sometimes algebraic or trigonometric identities can be used to rewrite the integrand so that the appropriate method is obvious. Example:

$$\int \sqrt{x}(1-\sqrt{x})dx = \int \sqrt{x} + x\,dx$$

2. **Look for Obvious Substitutions.** Try to find a function $u = g(x)$ in the integrand whose differential $du = g'(x)dx$ also appears. Example:

$$\int \frac{x}{x^2-1}dx$$

   We do not need trig-sub; $u = x^2 - 1$ will have $du = 2x\,dx$ and the u-sub is an easier integral.

3. **Classify the Integrand According to Its Form.** If Steps 1 and 2 have not led to a solution:

   (a) *Trigonometric functions.* If $f(x)$ is a product of powers of $\sin x, \cos x, \tan x, \sec x$, we use the techniques in Section 8.3.

   (b) *Rational functions.* If $f(x)$ is a rational function, then the integral should yield to the method of partial fractions, Section 8.5.

   (c) *Integration by parts.* If $f(x)$ is a product of two functions, and u-sub did not work, then integration by parts may be helpful. Remember the **LIPET** acronym.

   (d) *Other common substitutions.* Besides trig-substitutions, there are other common functions such that the substitution $x = g(t); dx = g'(t)\,dt$ reduce the integral in a meaningful way.

4. **Try Again.** If we still have not produced an answer, we have to think outside of the box.

   (a) *Try substitution.* Even if no substitution is obvious, some ingenuity can lead to a helpful substitution.

   (b) *Try parts.* We can always try integration by parts even if the integrand is not a product. Remember that integration of $\arcsin x$, and $\ln x$ both need integration by parts.

   (c) *Manipulate the integrand.* Algebraic or trigonometric manipulation beyond Part 1. may be needed. Example:

$$\int \frac{dx}{1-\cos x} = \int \frac{1}{1-\cos x}\frac{1+\cos x}{1+\cos x}dx = \int \frac{1+\cos x}{\sin^2 x}dx$$

   and our trig strategies will finish the integration.

   (d) *Relate the problem to previous problems.* You may recognize the integral from previous work. Example:

$$\int_0^1 \sqrt{1-x^2}dx$$

   can be computed as $\frac{\pi}{4}$ without any integration after realizing it a fourth of the area of a circle with radius 1.

   (e) *Use several methods.* Successfully applying one of the above methods will not necessarily produce any obvious integral. You may have to apply several different methods sequentially.

Let us take a closer look at strategy 3(d).

One common substitution when dealing with algebraic integrands including factors of $\sqrt[n]{g(x)}$ is to sub out a radical $u = \sqrt[n]{g(x)}$, $du = \dfrac{g'(x)}{n}(g(x))^{\frac{1}{n}-1}dx$. When $g(x)$ is a polynomial, this substitution can have the effect of transforming the integrand into a rational function to which we apply the method of partial fractions.

Example 8.6.1. $\displaystyle\int \frac{dx}{x-\sqrt{x+2}}$

Example 8.6.2. $\displaystyle\int \frac{dt}{t\sqrt{\ln t}}$

Other transcendental functions can be useful in substitutions. The hyperbolic functions $\sinh x, \cosh x$ are the functions that satisfy the relations

$$\cosh^2 x - \sinh^2 x = 1$$
$$\frac{d}{dx}\cosh x = \sinh x$$
$$\frac{d}{dx}\sinh x = \cosh x$$
$$\cosh(0) = 1; \sinh(0) = 0$$

The reciprocal and inverse functions for the hyperbolic functions are defined in the same way as for the trig functions: $\tanh x = \frac{\sinh x}{\cosh x}, \text{sech} x = \frac{1}{\cosh x}$, etc. We can see we have the derivative identities: $\dfrac{d}{dx}\tanh x = \text{sech}^2 x$ and $\dfrac{d}{dx}\text{sech } x = -\tanh x \text{ sech } x$.

The first identity above has a striking resemblance to the Pythagorean Identity, and from it we also see

$$1 - \tanh^2 x = \text{sech}^2 x.$$

Example 8.6.3. $\displaystyle\int \frac{dx}{1-x^2}$

Example 8.6.4. $\displaystyle\int \frac{dx}{x\sqrt{1+x^{2/3}}}$

Back to solving integrals in general.

Example 8.6.5. 1. $\displaystyle\int \frac{dx}{x\sqrt{1-x^2}}$

2. $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}} dx$

3. $\displaystyle\int \sin x \ln(\sin x)\, dx$

4. $\displaystyle\int_0^{\ln 2} \frac{1}{(1+e^x)^2} dx$

5. $\displaystyle\int x^2 \sinh x\, dx$

6. $\displaystyle\int_0^{\sqrt{2}/2} e^{\arcsin x}\,dx$

7. $\displaystyle\int e^x \sin^{998}(e^x)\cos^3(e^x)\,dx$

There are many integrals that are impossible to integrate analytically, primarily due to the integrands having no anti-derivative expressible in terms of elementary functions. Several of these occur frequently enough to warrant recognition, with numerical approximates well-known for certain special limits:

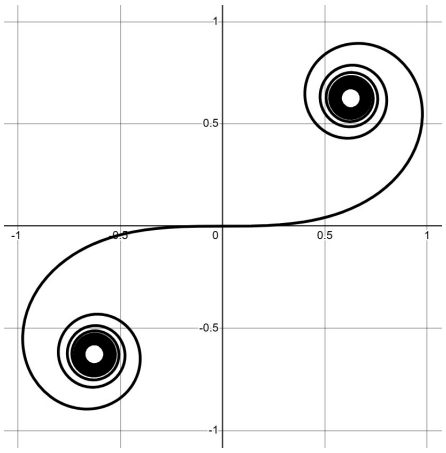
1. The error function appearing frequently in statistical applications:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

2. The Fresnel integrals find use in the study of optics:

$$S(x) = \int_0^x \sin t^2\, dt$$
$$C(x) = \int_0^x \cos t^2\, dt$$

   These two functions together parameterize the Euler spiral



3. The trigonometric integrals also appear in optics:

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt$$
$$\operatorname{Cin}(x) = \int_0^x \frac{1 - \cos t}{t} dt$$

4. The incomplete lower gamma function appears in the solution to all manner of nonequilibriaum transfer equations:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

5. The polylogarithm appear in quantum statistics and electrodynamics:

$$\mathrm{Li}_{s+1}(x) = \int_0^x \frac{\mathrm{Li}_s(t)}{t} dt$$
$$\text{where } \mathrm{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt$$

6. the logarithmic integral is used in number theory as a close approximation to the prime counting function as well as having uses in physics applications:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} haw$$